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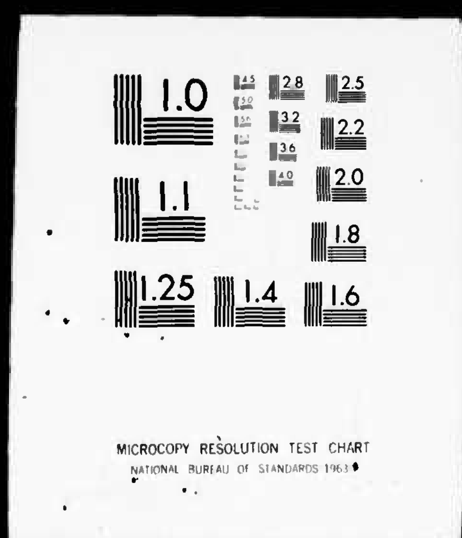

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SOME EXACT HYDRODYNAMIC FLOWS WITH FREE SURFACES

J. C. W. ROGERS

THE JOHNS HOPKINS UNIVERSITY ■ APPLIED PHYSICS LABORATORY

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Some exact solutions for the time-dependent incompressible irrotational flow of a fluid contained between two free surfaces are presented. The free surfaces are either concentric circular cylinders or concentric spheres. The cases where constant pressures are applied at one or the other of the free surfaces are also described, and the corresponding flows are analyzed for Taylor instability of the free boundaries. When the free boundaries are concentric circular cylinders, flows with constant non-zero circulation for circuits containing the cylinder axis are also described.

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SOME EXACT HYDRODYNAMIC FLOWS WITH FREE SURFACES

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CONTENTS

ABSTRACT

Some exact solutions for the time-dependent incompressible irrotational flow of a fluid contained between two free surfaces are presented. The free surfaces are either concentric circular cylinders or concentric spheres. The cases where constant pressures are applied at one or the other of the free surfaces are also described, and the corresponding flows are analyzed for Taylor instability of the free boundaries. When the free boundaries are concentric circular cylinders, flows with constant non-zero circulation for circuits containing the cylinder axis are also described.

1. INTRODUCTION

The exact hydrodynamic flows described in this report were found incident to carrying out the analytical phase of a project entitled "Ship-Wave Interactions" funded by the Office of Naval Research under Task No. NR 334-003. So far as we know, these exact solutions have not been reported elsewhere. We will say a little about our motivation for finding these solutions in the next paragraph, but our reasons for presenting them here in a separate report can be stated very simply: To put before the numerical hydrodynamics community some simple, but non-trivial, exact solutions of the non-linear equations for the motion of a liquid with a free surface, which might be used to check the accuracy of various numerical techniques for solving time-dependent hydrodynamic free boundary problems.

Our reason for introducing the flows described here is an outgrowth of a relation that has been found between turbulence and energy non-conservation for inviscid flows. Details of that connection will be reported elsewhere (Ref. 1). Briefly, for a hydrodynamic flow it turns out that a convenient measure of the local turbulence of the flow is given by the quantity

$$Q \equiv \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 u^2\right) + \nabla \cdot \left[u\left(\frac{1}{2}\rho_0 u^2 + P\right)\right], \tag{1.1}$$

where $u(x,t)$ is the velocity field, ρ_0 is the constant liquid density, and P is the pressure. The presence or absence of turbulence is characterized by whether or not $Q < 0$ (more precisely, in a stochastic framework, by whether or not $Q < 0$ "almost surely").

In the next paragraph, we will give an example of a flow with $u \cdot \vec{i} = u \cdot \vec{j} = 0$ and $u \cdot \vec{k}$ a function of z and t only, for which

$$Q \propto - \delta(z-z_0)\ \delta(t-t_0)\ . \tag{1.2a}$$

A natural question which arises in this: Is there a hydrodynamic flow with $u \cdot \vec{k} = 0$ and $u \cdot \vec{i}$, $u \cdot \vec{j}$ independent of z, such that

$$Q \propto - \delta(x-x_0)\ \delta(y-y_0)\ \delta(t-t_0)\ , \tag{1.2b}$$

and is there a flow with

$$Q \propto - \delta(x-x_0)\ \delta(y-y_0)\ \delta(z-z_0)\ \delta(t-t_0)\ \ ? \tag{1.2c}$$

Further, we may ask if somehow one can express general turbulent flows for different numbers of independent spatial variables in terms of superpositions of non-turbulent flows and elementary flows with turbulence characterized by (1.2a), (1.2b), or (1.2c). At this writing, we have not answered the latter inquiry, and indeed there will be no need to answer it until our analysis has proceeded to a more advanced stage. As regards the other question, in this report we construct flows having properties (1.2b) and (1.2c) as natural generalizations of the flow with Q satisfying (1.2a).

For a flow satisfying (1.2a), consider the hydrodynamic flow with initial conditions

$$\rho(\vec{x},0) = \begin{cases} 0 & 0 \le |z| < z_0 \\ \rho_0 & z_0 \le |z| \le z_1 \\ 0 & z_1 < |z| \end{cases} \tag{1.3a}$$

and

$$\vec{u}\,(\vec{x},0) = -\,U\,\vec{k}\,\mathrm{sgn}(z), \qquad z_0 \le |z| \le z_1 \; . \tag{1.3b}$$

Here $U > 0$. For $0 < t < \frac{z_0}{U}$, we have

$$\rho(\vec{x},t) = \begin{cases} 0 & 0 \le |z| < (z_0 - Ut) \\ \rho_0 & (z_0 - Ut) \le |z| \le (z_1 - Ut) \\ 0 & (z_1 - Ut) < |z| \end{cases} \tag{1.4a}$$

and

$$\vec{u}\,(\vec{x},t) = -\,U\,\vec{k}\,\mathrm{sgn}(z), \qquad (z_0 - Ut) \le |z| \le (z_1 - Ut). \tag{1.4b}$$

For later times, we choose the flow that satisfies

$$\rho(\vec{x},t) = \begin{cases} \rho_0 & 0 \le |z| \le (z_1 - z_0) \\ 0 & (z_1 - z_0) < |z| \end{cases}, \qquad t \ge \frac{z_0}{U} \tag{1.5a}$$

and

$$\vec{u}\,(\vec{x},t) = 0, \qquad 0 \le |z| \le (z_1 - z_0), \qquad t > \frac{z_0}{U} \; . \tag{1.5b}$$

For the flow given by (1.3)-(1.5), we readily find

$$Q = -\,\rho_0\,U^2\,(z_1 - z_0)\,\delta(z)\,\delta(t - \frac{z_0}{U}) \; . \tag{1.6}$$

This is a flow of type (1.2a); others may be found by shifting the origins of space and time.

Let us make the following observation: Instead of (1.5), one might choose the flow

$$\rho(\vec{x},t) = \begin{cases} 0 & 0 \le |z| < U(t - \frac{z_0}{U}) \\ \rho_0 & U(t - \frac{z_0}{U}) \le |z| \le z_1 - z_0 + U(t - \frac{z_0}{U}), \quad t \ge \frac{z_0}{U}, \\ 0 & z_1 - z_0 + U(t - \frac{z_0}{U}) < |z| \end{cases} \tag{1.7a}$$

and

$$\vec{u}\,(\vec{x},t) = U\,\vec{k}\,\mathrm{sgn}(z), \qquad U(t-\frac{z_0}{U}) \le |z| \le z_1 - z_0 + U(t-\frac{z_0}{U}), \qquad t > \frac{z_0}{U}\,. \tag{1.7b}$$

For the flow given by (1.3), (1.4), and (1.7), $Q \equiv 0$. We may think of the flow (1.3)-(1.5) as "perfectly inelastic", the flow (1.3), (1.4), (1.7) as "perfectly elastic". There are, of course, various intermediate possibilities. The reason we have focused on the flow (1.3)-(1.5) is that this is the flow that will be calculated by our water-wave algorithm (Ref. 2). That is not a mandatory feature of our calculation, but it seems like a good first approximation, and it is with that feature in mind that we have analyzed stability and turbulence for the flows generated by the algorithm. However, other numerical schemes may calculate instead the flow (1.3), (1.4), (1.7), or possibly intermediate cases. The exact flows we present in this report are not predicated on the assumption of inelasticity of "collisions" and will be of use whatever numerical scheme one may use. The only difference will be in how the various flows presented here are to be "patched together" at moments of "collision".

Like the flow (1.3)-(1.5), the flows to be presented in the next section have a free boundary consisting of two components unconnected to each other. Furthermore, they are "one-dimensional" in the sense that the velocities depend on only one spatial coordinate, if one uses cylindrical or spherical coordinates. In most of the cases, there is only one non-zero velocity component in an appropriate coordinate system. All the flows are irrotational as well as incompressible. As a consequence, it turns out that the "cylindrical" solutions have a greater variety than the "spherical" ones because of the possibility of having non-zero circulation for an irrotational flow in a multiply-connected, but not simply-connected, domain.

There are no rigid boundaries for the flows presented here, and in addition the domains occupied by the moving liquid are bounded. It has been seen (Ref. 3) that under such conditions the effect of gravity is only to bring about an acceleration of the whole flow configuration in the direction of the gravitational attraction. Thus, there is no loss of generality in ignoring the effect of gravity in these flows. If there is gravity present, the flows should be considered to be given in an appropriately accelerated system of coordinates. Of course, if one desires a more stringent test of his solution algorithms, not only may he compute the flow in the presence of gravity, but he may solve the equations in a coordinate system for which the special solutions given here do not possess such a simple symmetry. In addition to accelerated systems, such coordinates may be obtained by displacing the origin of coordinates and moving the system in uniform rectilinear motion. Since, however, the solutions given here are all relatively smooth, the demands made on a solution algorithm to duplicate them will be correspondingly limited, and the test on the algorithm afforded by comparison of the exact solution with the approximate one will be one of necessity, not of sufficiency.

2. DESCRIPTION OF THE FLOWS

For all the flows given here, the liquid occupies the domain

$$r_0(t) < r < r_1(t) \quad , \qquad (2.1)$$

where for three-dimensional flows $r = (x^2+y^2+z^2)^{\frac{1}{2}}$ and for two-dimensional flows $r = (x^2+y^2)^{\frac{1}{2}}$. The sets $\{\vec{x} \mid |\vec{x}| = r_0(t)\}$ and $\{\vec{x} \mid |\vec{x}| = r_1(t)\}$ are called the inner and outer free surface, respectively. The cases we consider are: (i) three-dimensional flow, zero pressure on the free surfaces; (ii) two-dimensional flow, zero pressure on the free surfaces; (iii) three-dimensional flow, pressure applied at one of the free surfaces; (iv) two-dimensional flow, pressure applied at one of the free surfaces; (v) two-dimensional flow with circulation, zero pressure on the free surfaces; and (vi) two-dimensional flow with circulation, pressure applied at one of the free surfaces.

We caution the reader that, for economy of notation, the same symbol may have different meanings in the descriptions of different flows. Of course, if an equation appearing in one flow description is referred to in another flow description, the symbols that appear therein will be consistent.

THREE-DIMENSIONAL FLOW, ZERO PRESSURE ON THE FREE SURFACES

Since the flow is incompressible and irrotational, the velocity field is derivable from a potential φ satisfying $\Delta\varphi = 0$. Bernoulli's equation is

$$\varphi_t + \frac{1}{2}(\nabla\varphi)^2 + \frac{P}{\rho_0} = 0 \quad , \qquad (2.2)$$

where P is the pressure and ρ_0 is the constant liquid density.

We write

$$\varphi = A(t)\left(\frac{1}{r_1(t)} - \frac{1}{r}\right) + B(t) \ . \tag{2.3}$$

There are four conditions to determine $A(t)$, $B(t)$, $r_0(t)$, and $r_1(t)$: The two kinematical conditions that points on the free surfaces remain there,

$$\dot{r}_0 = \frac{A}{r_0^2} \quad \text{and} \quad \dot{r}_1 = \frac{A}{r_1^2} \ , \tag{2.4}$$

and the two dynamical conditions that the pressure on the inner and outer free surfaces be zero. Using these pressure conditions we get, on substituting equation (2.3) into (2.2),

$$\dot{A}\left(\frac{1}{r_1} - \frac{1}{r_0}\right) - \frac{A}{r_1^2}\dot{r}_1 + \dot{B} + \frac{1}{2}\frac{A^2}{r_0^4} = 0 \tag{2.5}$$

and

$$-\frac{A}{r_1^2}\dot{r}_1 + \dot{B} + \frac{1}{2}\frac{A^2}{r_1^4} = 0 \ . \tag{2.6}$$

From the kinematical conditions (2.4), we get conservation of volume:

$$r_1^3(t) - r_0^3(t) = r_1^3(0) - r_0^3(0) = V \ . \tag{2.7}$$

By combining equations (2.4)-(2.6) we also get conservation of energy:

$$A^2(t)\left(\frac{1}{r_0(t)} - \frac{1}{r_1(t)}\right) = A^2(0)\left(\frac{1}{r_0(0)} - \frac{1}{r_1(0)}\right) = E \ . \tag{2.8}$$

It is easier to work with the equations in dimensionless form. In terms of E and V above we can find a characteristic time

$$t^* = \frac{V^{5/6}}{E^{1/2}} . \tag{2.9}$$

Letting

$$\tau = \frac{t}{t^*} , \quad R_0 = \frac{r_0}{V^{1/3}} , \quad R_1 = \frac{r_1}{V^{1/3}} , \quad \text{and}$$

$$\alpha = \frac{A}{E^{1/2} \; V^{1/6}} , \tag{2.10}$$

we get

$$R_1^3 - R_0^3 = 1 , \tag{2.11a}$$

$$\frac{1}{R_0} - \frac{1}{R_1} = \frac{1}{\alpha^2} , \tag{2.11b}$$

and

$$\frac{dV_1}{d\tau} = 3\alpha , \tag{2.11c}$$

where

$$V_1 = R_1^3 . \tag{2.11d}$$

If we use (2.11a,b,d) to express α in terms of V_1, we see that (2.11c) is an equation for τ in terms of V. Let us consider a special solution for which, at the origin of time, $\tau = 0$, we have $V_1(0) = 1$, or $R_0(0) = 0$. Then we have

$$\tau = \frac{1}{3} \int_1^{V_1} \frac{\left(1-(1-1/\xi)^{1/3}\right)^{1/2}}{(\xi-1)^{1/6}} \, d\xi . \tag{2.12}$$

There are two possible signs for the square roots in (2.12). We shall specialize to the case where the positive signs are taken.

We note the following limiting values of the integral in (2.12). When $V_1 \to 1$,

$$\tau \to \frac{2}{5} (V_1 - 1)^{5/6} \quad . \tag{2.13}$$

Equivalently,

$$V_1 \to \left(\frac{5}{2}\tau\right)^{6/5} + 1 \; , \tag{2.14a}$$

$$R_0 = (V_1 - 1)^{1/3} \to \left(\frac{5}{2}\tau\right)^{2/5} \quad . \tag{2.14b}$$

On the other hand, when $V_1 \to \infty$,

$$\tau \to \frac{1}{\sqrt{3}} V_1^{1/3} \quad , \tag{2.15a}$$

$$R_1 \to \sqrt{3}\,\tau \quad . \tag{2.15b}$$

In dimensional variables,

$$r_0 \to E^{1/5} \left(\frac{5}{2} t\right)^{2/5} \quad \text{as} \quad t \downarrow 0 \tag{2.16a}$$

and

$$r_1 \to \sqrt{3} \left(\frac{E}{V}\right)^{1/2} t \quad \text{as} \quad t \uparrow \infty \quad . \tag{2.16b}$$

Apart from limiting values, we see that (2.12) gives R_1 as a monotonically increasing function of τ, which we denote by

$$R_1 = F(\tau) \quad . \tag{2.17}$$

Now, suppose we have a general initial-value problem, with $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$ given. From (2.4), (2.7), and (2.8) we compute V and E. With $R_1(0)$ obtained from $r_1(0)$ and (2.10), we use (2.17) to compute a "time" $\tau_0 > 0$ such that

$$R_1(0) = F(\tau_0) \ . \tag{2.18}$$

If $\dot{r}_1(0) > 0$, the flow for $t > 0$ is given by

$$r_1(t) = V^{1/3} \ F\left(\tau_0 + \frac{t}{t^*}\right) , \tag{2.19}$$

where t^* is given in (2.9). $r_0(t)$ is found from (2.7). If $\dot{r}_1(0) < 0$, the flow for $0 < t < \tau_0 t^*$ is given by

$$r_1(t) = V^{1/3} \ F\left(\tau_0 - \frac{t}{t^*}\right) , \tag{2.20}$$

with $r_0(t)$ given again by (2.7). For $t > \tau_0 t^*$, the case of "inelastic collision" gives

$$r_1(t) = V^{1/3} , \tag{2.21a}$$

and in the case of "elastic collision"

$$r_1(t) = V^{1/3} \ F\left(\frac{t}{t^*} - \tau_0\right) . \tag{2.21b}$$

In the former case, with Q given by (1.1), we have

$$Q = - 2\pi \ \rho_0 \ E \ \delta(t-\tau_0 t^*) \ \delta(\vec{x}) \ . \tag{2.22}$$

In the latter case, $Q \equiv 0$.

TWO-DIMENSIONAL FLOW, ZERO PRESSURE ON THE FREE SURFACES

The similarities with the three-dimensional case are so extensive that we only indicate what changes need to be made in the equations above as we go along.

In place of (2.3) we have

$$\varphi = A(t)\, \ell n \frac{r}{r_1(t)} + B(t) \ . \tag{2.23}$$

The kinematical boundary conditions are

$$\dot{r}_0 = \frac{A}{r_0} \quad \text{and} \quad \dot{r}_1 = \frac{A}{r_1} \ , \tag{2.24}$$

from which we get conservation of volume,

$$r_1^2(t) - r_0^2(t) = r_1^2(0) - r_0^2(0) = V \ . \tag{2.25}$$

The conservation of energy is expressed by

$$A^2(t)\, \ell n \frac{r_1(t)}{r_0(t)} = A^2(0)\, \ell n \frac{r_1(0)}{r_0(0)} = E \ . \tag{2.26}$$

The characteristic time is

$$t^* = \frac{V}{E^{1/2}} \quad . \tag{2.27}$$

Let

$$\tau = \frac{t}{t^*}\ , \quad R_0 = \frac{r_0}{V^{1/2}}\ , \quad R_1 = \frac{r_1}{V^{1/2}}\ , \quad \alpha = \frac{A}{E^{1/2}} \quad . \tag{2.28}$$

Then the equations assume the dimensionless form

$$R_1^2 - R_0^2 = 1 \ , \tag{2.29a}$$

$$\ell n \frac{R_1}{R_0} = \frac{1}{\alpha^2} \ , \tag{2.29b}$$

$$\frac{dV_1}{d\tau} = 2\alpha \ , \tag{2.29c}$$

where

$$V_1 = R_1^2 \ . \tag{2.29d}$$

Consider the flow for which $V_1(0) = 1$, $R_0(0) = 0$. In place of (2.12) we get

$$\tau = \frac{1}{2\sqrt{2}} \int_1^{V_1} \left(\ell n \left(1 + \frac{1}{\xi - 1} \right) \right)^{\frac{1}{2}} d\xi \ , \tag{2.30}$$

where, as above, we take the positive sign for the square root. This gives R_1 as a monotonically increasing function of τ, denoted by

$$R_1 = F(\tau) \ . \tag{2.31}$$

The following limits are of interest. When $V_1 \to 1$,

$$\tau \to \frac{1}{2\sqrt{2}} \left(\ell n \frac{1}{V_1 - 1} \right)^{\frac{1}{2}} (V_1 - 1) \ , \tag{2.32}$$

or

$$V_1 \to 1 + \frac{2\sqrt{2}\,\tau}{\sqrt{\ell n \frac{1}{\tau}}} \ , \tag{2.33a}$$

$$R_0 \to \frac{2^{3/4} \ \tau^{1/2}}{\left(\ell n \frac{1}{\tau} \right)^{1/4}} \ . \tag{2.33b}$$

When $V_1 \to \infty$,

$$\tau \to \frac{1}{\sqrt{2}} V_1^{1/2} , \tag{2.34a}$$

$$R_1 \to \sqrt{2}\, \tau . \tag{2.34b}$$

In dimensional variables,

$$r_0 \to \frac{2^{3/4} \quad E^{1/4} \quad t^{1/2}}{\left(\ell n\left(\frac{V}{E^{1/2}\ t}\right)\right)^{1/4}} \quad \text{as} \quad t \downarrow 0 \tag{2.35a}$$

and

$$r_1 \to \sqrt{2}\left(\frac{E}{V}\right)^{\frac{1}{2}} t \quad \text{as} \quad t \uparrow \infty . \tag{2.35b}$$

Suppose, in general, we are given $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$. From (2.24)-(2.27) we compute V, E, and t^*. Find $R_1(0)$ by (2.28), and then use (2.31) to find $\tau_0 > 0$ such that

$$R_1(0) = F(\tau_0) . \tag{2.36}$$

For $\dot{r}_1(0) > 0$, the flow for $t > 0$ is

$$r_1(t) = V^{1/2} \quad F\left(\tau_0 + \frac{t}{t^*}\right) . \tag{2.37}$$

For $\dot{r}_1(0) < 0$, the flow for $0 < t < t^* \tau_0$ is

$$r_1(t) = V^{1/2} \quad F\left(\tau_0 - \frac{t}{t^*}\right) . \tag{2.38}$$

For $t > t^* \tau_0$ and $\dot{r}_1(0) < 0$, we get

$$r_1(t) = V^{1/3} \tag{2.39a}$$

in the inelastic case, and

$$r_1(t) = V^{1/3} F\left(\frac{t}{t^*} - \tau_0\right) \tag{2.39b}$$

in the elastic case. In the former case,

$$Q = -\pi \rho_0 E \delta(t - t^* \tau_0) \delta(\vec{x}) . \tag{2.40}$$

THREE-DIMENSIONAL FLOW, PRESSURE APPLIED AT ONE OF THE FREE SURFACES

The velocity potential is given by (2.3), and the kinematical conditions at the free surface are still given by (2.4). Since the only quantity of interest is the difference between the pressures applied at the inner and outer surfaces, we have no loss of generality in setting the pressure at the inner surface equal to zero. Let the pressure at the outer surface have the value P_1 constant on the surface. Then equation (2.5) still holds, but (2.6) should be replaced by

$$-\frac{A}{r_1^2}\dot{r}_1 + \dot{B} + \frac{1}{2}\frac{A^2}{r_1^4} + \frac{P_1}{\rho_0} = 0 . \tag{2.41}$$

The conservation of volume is expressed by (2.7). The conservation of energy takes the following form when P_1 is independent of the time:

$$A^2(t)\left(\frac{1}{r_0(t)} - \frac{1}{r_1(t)}\right) + \frac{2}{3}\frac{P_1}{\rho_0} r_0^3(t) = A^2(0)\left(\frac{1}{r_0(0)} - \frac{1}{r_1(0)}\right) + \frac{2}{3}\frac{P_1}{\rho_0} r_0^3(0) \equiv \epsilon . \tag{2.42}$$

Consider the case when ϵ, as defined by (2.42), is > 0. This is the case whenever $P_1 > 0$ and sometimes when $P_1 < 0$. Then let us write $\epsilon = E$. Define t^*, τ, R_0, R_1, α, and V_1 by (2.9), (2.10), and (2.11d). Define π_1 by

$$\pi_1 = \frac{V}{E} \frac{P_1}{\rho_0} \quad . \tag{2.43}$$

Then the governing equations are (2.11a), (2.11c), and

$$\alpha^3 \left(\frac{1}{R_0} - \frac{1}{R_1}\right) = 1 - \frac{2}{3} \pi_1 \ (R_1^3 - 1) \ . \tag{2.44}$$

For the flow with $V_1(0) = 1$ ($R_0(0) = 0$)

$$\tau = \frac{1}{3} \int_1^{V_1} \frac{\left(1 - \left(1 - \frac{1}{\xi}\right)^{1/3}\right)^{1/2}}{(\xi - 1)^{1/6} \left(1 - \frac{2}{3} \pi_1 \ (\xi - 1)\right)^{1/2}} \, d\xi \quad . \tag{2.45}$$

We take the positive signs for the square roots. The behavior for small time ($\tau \downarrow 0$) is given by (2.14a,b).

Two cases should be distinguished. In the first, $P_1 > 0$. In that case the expression (2.45) for τ fails for

$$V_1 > \xi_0 = 1 + \frac{3}{2\pi_1} \quad . \tag{2.46}$$

Define τ^* by

$$\tau^* = \frac{1}{3}\int_1^{1+\frac{3}{2\pi_1}} \frac{\left(1-\left(1-\frac{1}{\xi}\right)^{1/3}\right)^{1/2}}{(\xi-1)^{1/6}\left(1-\frac{2}{3}\pi_1(\xi-1)\right)^{1/2}}\, d\xi \; . \tag{2.47}$$

Then as $(\tau^*-\tau) \downarrow 0$, we get

$$V_1 \to 1 + \frac{3}{2\pi_1} - \frac{3}{2}\pi_1 \frac{\left(1+\frac{3}{2\pi_1}\right)^{1/3}\left(\frac{3}{2\pi_1}\right)^{1/3}}{\left(1+\frac{3}{2\pi_1}\right)^{1/3} - \left(\frac{3}{2\pi_1}\right)^{1/3}} (\tau^*-\tau)^2 \tag{2.48a}$$

and

$$R_0 \to \frac{3}{2\pi_1}^{1/3} - \frac{1}{2}\frac{\pi_1\left(1+\frac{3}{2\pi_1}\right)^{1/3}}{\left(\frac{3}{2\pi_1}\right)^{1/3}\left(\left(1+\frac{3}{2\pi_1}\right)^{1/3} - \left(\frac{3}{2\pi_1}\right)^{1/3}\right)} (\tau^* - \tau)^2 \; . \tag{2.48b}$$

For $0 < \tau < \tau^*$, (2.45) gives R_1 as a monotonically increasing function of τ, which we denote by

$$R_1 = F(\tau) \; . \tag{2.49}$$

For the general initial-value problem, we have $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$. From these, by (2.4), (2.7), (2.42), (2.9), (2.43), and (2.47), we may calculate V, E, t^*, π_1, and τ^*. If $\pi_1 > 0$, there will be a $\tau_0 \in (0,\tau^*]$ such that

$$R_1(0) = F(\tau_0) \; .$$

When $\dot{r}_1(0) > 0$, the flow for $0 \le t \le t^*(\tau^*-\tau_0)$ is

$$r_1(t) = V^{1/3}\; F\left(\tau_0 + \frac{t}{t^*}\right) , \tag{2.50a}$$

for $t^*(\tau^*-\tau_0) \le t < t^*(2\tau^*-\tau_0)$ it is

$$r_1(t) = V^{1/3}\ F\left(2\tau^*-\tau_0-\frac{t}{t^*}\right)\ , \tag{2.50b}$$

and for $t > t^*(2\tau^*-\tau_0)$ it is

$$r_1(t) = V^{1/3} \tag{2.50c}$$

in the inelastic case. When $\dot{r}_1(0) < 0$, the flow for $0 \le t < t^*\tau_0$ is

$$r_1(t) = V^{1/3}\ F\left(\tau_0-\frac{t}{t^*}\right)\ , \tag{2.51a}$$

and for $t > t^*\tau_0$ it is

$$r_1(t) = V^{1/3} \tag{2.51b}$$

in the inelastic case. In the case of elastic collisions, these flows will be time-periodic with period $2t^*\tau^*$, and $r_0(t)$ will vary between 0 and $V^{1/3}\left(\frac{3}{2\Pi_1}\right)^{1/3}$.

At this juncture, let us observe that one or the other of the free surfaces will tend to exhibit Taylor instability at some point during the flow, and hence the solutions with non-zero pressures imposed on the free surfaces will be of rather limited physical interest. It follows from an analysis elsewhere (Ref. 3) that we may expect Taylor instability at the surface $r = r_0(t)$ if and only if

$$\left.\frac{\partial P}{\partial r}\right|_{r_0(t)} < 0 \tag{2.52a}$$

and Taylor instability at the surface $r = r_1(t)$ if and only if

$$\left.\frac{\partial P}{\partial r}\right|_{r_1(t)} > 0 \; . \qquad (2.52b)$$

Note that we can write

$$P = \theta + P_1 \frac{\dfrac{1}{r_0(t)} - \dfrac{1}{r}}{\dfrac{1}{r_0(t)} - \dfrac{1}{r_1(t)}} \; , \qquad (2.53a)$$

where θ is superharmonic and $\theta = 0$ at $r_0(t)$ and $r_1(t)$. Hence,

$$\left.\frac{\partial \theta}{\partial r}\right|_{r_0(t)} \geq 0 \quad \text{and} \quad \left.\frac{\partial \theta}{\partial r}\right|_{r_1(t)} \leq 0 \; . \qquad (2.53b)$$

Accordingly, from (2.52) and (2.53), we will never get Taylor instability at $r = r_0(t)$ when $P_1 \geq 0$, or at $r = r_1(t)$ when $P_1 \leq 0$.

For the case we have considered so far, where $P_1 > 0$, if $\dot{r}_1(0) > 0$ then, because $r_1(t)$ is bounded above by $V^{1/3}\left(1+\frac{3}{2\pi_1}\right)^{1/3}$, we must get $\ddot{r}_1(t) < 0$ for some $t \in (0, t^*(\tau^*-\tau_0))$, and thus Taylor instability at $r = r_1(t)$. An explicit calculation, based on (2.2), (2.3), (2.44), and the known values of P at $r = r_0(t)$ and $r = r_1(t)$, yields

$$\frac{V^{4/3}}{E}\,\frac{1}{\rho_0}\left.\frac{\partial P}{\partial r}\right|_{r_1(t)} = \frac{1}{R_1^2\left(\frac{1}{R_0}-\frac{1}{R_1}\right)}\left\{\pi_1 - \left[1-\frac{2}{3}\pi_1(R_1^3-1)\right]\frac{1}{2}\left(\frac{1}{R_0}-\frac{1}{R_1}\right)\left(\frac{1}{R_0^2}+\frac{2}{R_0R_1}+\frac{3}{R_1^2}\right)\right\} . \qquad (2.54)$$

Define R_0^* as the unique positive solution of

$$\pi_1 = \frac{3}{2}\,\frac{((R_0^{*3}+1)^{1/3} - R_0^*)^2}{R_0^{*3}\,(4(R_0^{*3}+1)^{1/3} - 3R_0^*)}\left((R_0^{*3}+1)^{2/3} + 2R_0^*\,(R_0^{*3}+1)^{1/3} + 3R_0^{*2}\right) . \qquad (2.55)$$

It is easy to see that $0 < R_0^* < \left(\frac{3}{2\pi_1}\right)^{1/3}$. For $r_0(t) > V^{1/3} R_0^*$, we get Taylor instability at $r_1(t)$. If $\dot{r}_1(0) < 0$, then we do not get Taylor instability at the boundary $r = r_1(t)$ unless $r_0(0) > V^{1/3}\ R_0^*$.

Consider the case where $P_1 < 0$ but ϵ in (2.42) is > 0. We may still use equation (2.45). Define

$$\tau^+ = \frac{1}{3}\int_1^\infty \frac{\left(1-\left(1-\frac{1}{\xi}\right)^{1/3}\right)^{1/2}}{(\xi-1)^{1/6}\left(1-\frac{2}{3}\pi_1(\xi-1)\right)^{1/2}}\,d\xi\ . \tag{2.56}$$

Then we will have $V_1 \uparrow \infty$ as $t \uparrow t^* \tau^+$. This follows also from the equation of motion

$$\ddot{r}_1 = -\frac{1}{\rho_0}\left.\frac{\partial P}{\partial r}\right|_{r_1} \rightarrow -\frac{3P_1}{\rho_0}\frac{r_1^2}{V} \quad \text{as} \quad r_1 \uparrow \infty\ . \tag{2.57}$$

For $0 < \tau < \tau^+$, (2.45) gives R_1 as a monotonically increasing function of τ:

$$R_1 = F(\tau)\quad . \tag{2.58}$$

In the same way we derived equation (2.54), we find

$$\frac{V^{4/3}}{E}\frac{1}{\rho_0}\left.\frac{\partial P}{\partial r}\right|_{r_0(t)} = \frac{1}{R_0^2\left(\frac{1}{R_0}-\frac{1}{R_1}\right)}\left\{\pi_1 + \left[1-\frac{2}{3}\pi_1(R_1^3-1)\right]\frac{1}{2}\left(\frac{1}{R_0}-\frac{1}{R_1}\right)\left(\frac{3}{R_0^2}+\frac{2}{R_0R_1}+\frac{1}{R_1^2}\right)\right\}\ . \tag{2.59}$$

It is easy to see that there is a unique positive solution R_0^+ of

$$-\pi_1 = \frac{3}{2}\frac{\left((R_0^{+3}+1)^{1/3}-R_0^+\right)^2}{R_0^{+4}}\left(3(R_0^{+3}+1)^{2/3}+2R_0^+(R_0^{+3}+1)^{1/3}+R_0^{+2}\right)\ . \tag{2.60}$$

For $r_0(t) > V^{1/3} R_0^+$, we get Taylor instability at $r_0(t)$.

Now consider the general initial-value problem, where we are given $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$. From (2.4), (2.7), (2.42), (2.9), (2.43), and (2.56) we compute V, E, t^*, Π_1, and τ^+. We find $\tau_0 \in (0,\tau^+)$ as the solution of

$$R_1 = F(\tau_0) \ . \tag{2.61}$$

If $\dot{r}_1(0) > 0$, the flow for $0 \le t < t^* (\tau^+ - \tau_0)$ is given by

$$r_1(t) = V^{1/3} \ F\left(\tau_0 + \frac{t}{t^*}\right) . \tag{2.62}$$

For $t \ge t^* (\tau^+ - \tau_0)$ the flow is undefined. If $\dot{r}_1(0) < 0$, the flow for $0 \le t < t^* \tau_0$ is given by

$$r_1(t) = V^{1/3} \ F\left(\tau_0 - \frac{t}{t^*}\right) . \tag{2.63a}$$

In the inelastic case, for $t > t^* \tau_0$ we have

$$r_1(t) = V^{1/3} , \tag{2.63b}$$

and in the elastic case, for $t^* \tau_0 < t < t^* (\tau_0 + \tau^+)$ we have

$$r_1(t) = V^{1/3} \ F\left(\frac{t}{t^*} - \tau_0\right) . \tag{2.63c}$$

Suppose finally that ϵ as given by (2.42) is < 0. We write $E = -\epsilon$. Define t^*, τ, R_0, R_1, α, and V_1 by (2.9), (2.10), and (2.11d), and let

$$\Pi_1 = \frac{V}{E}\,\frac{P_1}{\rho_0} \quad . \tag{2.64}$$

The equations of motion in dimensionless form are given by (2.11a), (2.11c), and

$$\alpha^2\left(\frac{1}{R_0} - \frac{1}{R_1}\right) = -1 - \frac{2}{3}\Pi_1\left(R_1^3 - 1\right) \quad . \tag{2.65}$$

It is clear that for this flow, for all time,

$$R_0^3 \geq -\frac{3}{2\Pi_1} \quad . \tag{2.66}$$

Thus, the liquid remains away from the origin, and we do not have the possibility of "collisions", either elastic or inelastic.

Consider the flow with $V_1(0) = -\frac{3}{2\Pi_1} + 1$:

$$\tau = \frac{1}{3}\int_{-\frac{3}{2\Pi_1}+1}^{V_1} \frac{\left(1-\left(1-\frac{1}{\xi}\right)^{1/3}\right)^{1/2}}{(\xi-1)^{1/6}\left(-1-\frac{2}{3}\Pi_1(\xi-1)\right)^{1/2}}\,d\xi \quad . \tag{2.67}$$

We take positive signs for the square roots. (2.67) gives R_1 as a monotone increasing function of τ:

$$R_1 = F(\tau) \, . \tag{2.68}$$

As $\tau \downarrow 0$ we get from (2.67)

$$V_1 \to 1 - \frac{3}{2\Pi_1} + \frac{9}{4}\,\frac{\left(1-\frac{3}{2\Pi_1}\right)^{1/3}}{\left(-\frac{3}{2\Pi_1}\right)^{2/3}\left(\left(1-\frac{3}{2\Pi_1}\right)^{1/3} - \left(-\frac{3}{2\Pi_1}\right)^{1/3}\right)}\,\tau^2 \tag{2.69a}$$

and

$$R_0 \to \left(-\frac{3}{2\pi_1}\right)^{1/3} - \frac{\frac{1}{2}\pi_1\left(1-\frac{3}{2\pi_1}\right)^{1/3}}{\left(-\frac{3}{2\pi_1}\right)^{1/3}\left(\left(1-\frac{3}{2\pi_1}\right)^{1/3}-\left(-\frac{3}{2\pi_1}\right)^{1/3}\right)}\tau^2 . \tag{2.69b}$$

Define τ^+ by

$$\tau^+ = \frac{1}{3}\int_{-\frac{3}{2\pi_1}+1}^{\infty} \frac{\left(1-\left(1-\frac{1}{\xi}\right)^{1/3}\right)^{1/2}}{(\xi-1)^{1/6}\left(-1-\frac{2}{3}\pi_1(\xi-1)\right)^{1/2}}\,d\xi . \tag{2.70}$$

Using (2.2), (2.3), (2.65), and the prescribed values of P at the free surfaces, we get

$$\frac{V^{4/3}}{E}\frac{1}{\rho_0}\left.\frac{\partial P}{\partial r}\right|_{r_0(t)}$$

$$= \frac{1}{R_0^2\left(\frac{1}{R_0}-\frac{1}{R_1}\right)}\left\{\pi_1\left(1-\frac{1}{R_1^3}\,\frac{1+\frac{2}{3}\frac{R_0}{R_1}+\frac{1}{3}\frac{R_0^2}{R_1^2}}{1+\frac{R_0}{R_1}+\frac{R_0^2}{R_1^2}}\right)-\frac{1}{2}\left(\frac{1}{R_0}-\frac{1}{R_1}\right)\left(\frac{3}{R_0^2}+\frac{2}{R_0R_1}+\frac{1}{R_1^2}\right)\right\} . \tag{2.71}$$

It is clear from (2.71) that we get Taylor instability at the surface $r = r_0(t)$ throughout this flow.

For the general initial-value problem, we are given $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$. Using (2.4), (2.7), (2.42), the equation $E = -\epsilon$, (2.9), (2.64), and (2.70), we compute V, E, t^*, π_1, and τ^+. We find $\tau_0 \in [0,\tau^+)$ as the solution of

$$R_1 = F(\tau_0) . \tag{2.72}$$

If $\dot{r}_1(0) > 0$, the flow for $0 \le t < t^* (\tau^+ - \tau_0)$ is

$$r_1(t) = V^{1/3}\ F\left(\tau_0 + \frac{t}{t^*}\right) . \tag{2.73}$$

The flow is undefined for $t \ge t^* (\tau^+ - \tau_0)$. If $\dot{r}_1(0) < 0$, for $0 \le t \le t^* \tau_0$, we have

$$r_1(t) = V^{1/3}\ F\left(\tau_0 - \frac{t}{t^*}\right) \tag{2.74a}$$

and for $t^* \tau_0 \le t < t^* (\tau_0 + \tau^+)$,

$$r_1(t) = V^{1/3}\ F\left(\frac{t}{t^*} - \tau_0\right) . \tag{2.74b}$$

For $t \ge t^* (\tau_0 + \tau^+)$ the flow is undefined.

TWO-DIMENSIONAL FLOW, PRESSURE APPLIED AT ONE OF THE FREE SURFACES

The velocity potential is given by (2.23) and the kinematical free surface conditions by (2.24). We set the pressure on the inner surface equal to 0 and assume that the pressure assumes a value P_1 that is constant over the outer surface $r = r_1$. The conservation of volume is given by (2.25). When P_1 is constant in time, the conservation of energy takes the form

$$A^2(t)\ \ln \frac{r_1(t)}{r_0(t)} + \frac{P_1}{\rho_0}\ r_0^2(t) = A^2(0)\ \ln \frac{r_1(0)}{r_0(0)} + \frac{P_1}{\rho_0}\ r_0^2(0) \equiv \epsilon . \tag{2.75}$$

ϵ as defined here will be > 0 whenever $P_1 > 0$, and sometimes when $P_1 < 0$. When $\epsilon > 0$, we set $E = \epsilon$. We define t^*, τ, R_0, R_1, α, and V_1

by (2.27), (2.28), and (2.29d). Let

$$\Pi_1 = \frac{V}{E}\,\frac{P_1}{\rho_0} \quad . \tag{2.76}$$

The equations governing the flow are (2.29a), (2.29c), and

$$\alpha^2 \ \ln \frac{R_1}{R_0} = 1 - \Pi_1 (R_1^2 - 1) \ . \tag{2.77}$$

Let us examine the special flow with $V_1(0) = 1$ ($R_0(0) = 0$):

$$\tau = \frac{1}{2\sqrt{2}} \int_1^{V_1} \frac{\left(\ln\left(1+\frac{1}{\xi-1}\right)\right)^{1/2}}{\left(1-\Pi_1(\xi-1)\right)^{1/2}}\, d\xi \ . \tag{2.78}$$

As usual, we take positive signs for all square roots. The behavior for small τ is given by (2.33a,b).

Suppose $P_1 > 0$. Then it is clear from the energy equation (2.77) that V_1 will have an upper bound,

$$\xi_0 = 1 + \frac{1}{\Pi_1} \quad . \tag{2.79}$$

We define τ^* by

$$\tau^* = \frac{1}{2\sqrt{2}} \int_1^{1+\frac{1}{\Pi_1}} \frac{\left(\ln\left(1+\frac{1}{\xi-1}\right)\right)^{1/2}}{\left(1-\Pi_1(\xi-1)\right)^{1/2}}\, d\xi \ . \tag{2.80}$$

As $(\tau^*-\tau) \downarrow 0$, we get

$$V_1 \to 1 + \frac{1}{\Pi_1} - \frac{2\Pi_1}{\ln(1+\Pi_1)}\ (\tau^*-\tau)^2 \tag{2.81a}$$

and

$$R_0 \to \left(\frac{1}{\pi_1}\right)^{1/2} - \frac{(\pi_1)^{3/2}}{\ell n(1+\pi_1)} (\tau^* - \tau)^2 \quad . \tag{2.81b}$$

For $0 < \tau < \tau^*$, (2.78) gives R_1 as a monotonically increasing function of τ, which we denote by

$$R_1 = F(\tau) \quad . \tag{2.82}$$

As we have seen in the three-dimensional case, if we get Taylor instability at all it will occur at the surface $r = r_1(t)$. Using (2.2), (2.23), (2.77), and the prescribed values of the pressure on the free surfaces, we get

$$\frac{V^{3/2}}{E} \frac{1}{\rho_0} \left.\frac{\partial P}{\partial r}\right|_{r_1(t)}$$

$$= \frac{1}{R_1 \ell n \frac{R_1}{R_0}} \left\{ \pi_1 - [1 - \pi_1 (R_1^2 - 1)] \left(\frac{\frac{1}{R_0^2} - \frac{1}{R_1^2}}{\ell n \frac{R_1^2}{R_0^2}} - \frac{1}{R_1^2} \right) \right\} \quad . \tag{2.83}$$

We can check that there is only one positive root R_0^* of the equation

$$-\pi_1 = \frac{\ell n \left(1 + \frac{1}{R_0^{*2}}\right) - \frac{1}{R_0^{*2}}}{\ell n \left(1 + \frac{1}{R_0^{*2}}\right) + 1} \tag{2.84}$$

and this root satisfies $0 < R_0^* < \left(\frac{1}{\pi_1}\right)^{1/2}$.

Now let us consider the general initial-value problem, where we are given $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$. Using (2.24), (2.25), (2.75), the relation

$E = \epsilon$, (2.27), (2.76), and (2.80), we find V, E, t^*, Π_1, and τ^*. There will be a $\tau_0 \in (0,\tau^*]$ with

$$R_1(0) = F(\tau_0) .$$

If $\dot{r}_1(0) > 0$, the flow for $0 \le t \le t^* (\tau^* - \tau_0)$ is

$$r_1(t) = V^{1/2} \; F\left(\tau_0 + \frac{t}{t^*}\right), \tag{2.85a}$$

for $t^* (\tau^* - \tau_0) \le t < t^* (2\tau^* - \tau_0)$, it is

$$r_1(t) = V^{1/2} \; F\left(2\tau^* - \tau_0 - \frac{t}{t^*}\right), \tag{2.85b}$$

and for $t > t^* (2\tau^* - \tau_0)$ it is

$$r_1(t) = V^{1/2} \tag{2.85c}$$

in the inelastic case. In the elastic case, for $t^* (2\tau^* - \tau_0) < t \le t^* (3\tau^* - \tau_0)$, we get

$$r_1(t) = V^{1/2} \; F\left(\frac{t}{t^*} - 2\tau^* + \tau_0\right) \tag{2.85d}$$

and a flow that is periodic in time with period $2\tau^* t^*$. For these flows we will have Taylor instability at $r = r_1(t)$ whenever $r_0(t) > V^{1/2} R_0^*$, which will eventually occur if $\dot{r}_1(0) > 0$. If $\dot{r}_1(0) < 0$, the flow for $0 \le t < t^* \tau_0$ is

$$r_1(t) = V^{1/2} \; F\left(\tau_0 - \frac{t}{t^*}\right) \tag{2.86a}$$

and for $t > t^* \tau_0$ it is

$$r_1(t) = V^{1/2} \tag{2.86b}$$

in the inelastic case. In the elastic case, for $t^* \tau_0 < t \leq t^* (\tau_0 + \tau^*)$, we have

$$r_1(t) = V^{1/2} F\left(\frac{t}{t^*} - \tau_0\right) , \tag{2.86c}$$

and for $t^* (\tau_0 + \tau^*) \leq t < t^* (\tau_0 + 2\tau^*)$, we have

$$r_1(t) = V^{1/2} F\left(2\tau^* + \tau_0 - \frac{t}{t^*}\right) , \tag{2.86d}$$

the flow being periodic in time with period $2t^* \tau^*$. In the inelastic case we will not get Taylor instability at $r = r_1(t)$ if $\dot{r}_1(0) < 0$ and $r_0(0) \leq V^{1/2} R_0^*$. Otherwise, at some point in the flow, Taylor instability will occur on the outer surface.

Next we suppose that ϵ in (2.75) is > 0, but $P_1 < 0$. Equation (2.78) still holds and gives R_1 as a monotonically increasing function of τ:

$$R_1 = F(\tau) , \tag{2.87}$$

for $0 \leq \tau < \infty$. From (2.78) we see that, when $V_1 \uparrow \infty$,

$$\tau \rightarrow \frac{1}{2\sqrt{2}} \left(-\frac{1}{\Pi_1}\right)^{1/2} \ln V_1 \tag{2.88a}$$

or

$$R_1 \rightarrow e^{\sqrt{2}\, (-\Pi_1)^{1/2}\, \tau} . \tag{2.88b}$$

We can also verify this directly from the equation of motion,

$$\ddot{r}_1 = -\frac{1}{\rho_o}\left.\frac{\partial P}{\partial r}\right|_{r_1} \rightarrow -\frac{2}{\rho_o V} P_1 r_1 \quad \text{as} \quad r_1 \uparrow \infty . \tag{2.89}$$

As we have seen, Taylor instability can only occur at the inner boundary. By using (2.2), (2.23), (2.77), and the prescribed values of P at the boundaries, we find that

$$\frac{V^{3/2}}{E}\frac{1}{\rho_o}\left.\frac{\partial P}{\partial r}\right|_{r_o(t)}$$

$$= \frac{1}{R_o \ln \frac{R_1}{R_o}}\left\{\pi_1 + [1-\pi_1(R_1^2-1)]\left(\frac{1}{R_o^2} - \left(\frac{1}{R_o^2} - \frac{1}{R_1^2}\right)\frac{1}{\ln \frac{R_1^2}{R_o^2}}\right)\right\} . \tag{2.90}$$

There is a unique positive solution, R_o^+, of the equation

$$\left(1+\frac{1}{R_o^{+2}}\right) \ln\left(1+\frac{1}{R_o^{+2}}\right) - \frac{1}{R_o^{+2}} = -\pi_1 . \tag{2.91}$$

For $r_o(t) > V^{1/2} R_o^+$, we will get Taylor instability at $r = r_o(t)$.

In the general initial-value problem we are given $r_o(0)$, $r_1(0)$, and $\dot{r}_1(0)$. We use (2.24), (2.25), (2.75), the equation $E = \epsilon$, (2.27), and (2.76) to find V, E, t^*, and π_1. We find $\tau_o \in (0,\infty)$ such that

$$R_1(0) = F(\tau_o) .$$

If $\dot{r}_1(0) > 0$, the solution for $t \geq 0$ is

$$r_1(t) = V^{1/2} F\left(\tau_o + \frac{t}{t^*}\right) . \tag{2.92}$$

If $\dot{r}_1(0) < 0$, the solution for $0 \le t < t^* \tau_0$ is

$$r_1(t) = V^{1/2} \, F\left(\tau_0 - \frac{t}{t^*}\right) , \tag{2.93a}$$

and for $t > t^* \tau_0$ it is

$$r_1(t) = V^{1/2} \tag{2.93b}$$

in the inelastic case. In the elastic case, for $t > t^* \tau_0$, we have

$$r_1(t) = V^{1/2} \, F\left(\frac{t}{t^*} - \tau_0\right) . \tag{2.93c}$$

There will be no Taylor instability in the inelastic case if $\dot{r}_1(0) < 0$ and $r_0(0) \le V^{1/2} R_0^+$. Otherwise, at some point in the flow, we will get Taylor instability on the inner surface.

Now suppose ϵ in (2.75) is < 0. We write $E = -\epsilon$. We use (2.27), (2.28), and (2.29d) to define t^*, τ, R_0, R_1, α, and V_1, and we let

$$\pi_1 = \frac{V}{E} \, \frac{P_1}{\rho_0} . \tag{2.94}$$

In dimensionless variables, the equations of motion are (2.29a), (2.29c), and

$$\alpha^2 \, \ell n \frac{R_1}{R_0} = -1 - \pi_1 (R_1^2 - 1) . \tag{2.95}$$

We must have $R_0^2 \ge -\frac{1}{\pi_1}$, and thus we cannot have "collisions" of liquid at the origin.

Let us examine the special flow with $V_1(0) = 1 - \frac{1}{\pi_1}$. We get

$$\tau = \frac{1}{2\sqrt{2}} \int_{1-\frac{1}{\pi_1}}^{V_1} \frac{\left(\ln\left(1+\frac{1}{\xi-1}\right)\right)^{1/2}}{(-1-\pi_1(\xi-1))^{1/2}} d\xi , \tag{2.96}$$

and taking positive signs for the square roots, this gives R_1 as an increasing function of $\tau \in [0,\infty)$:

$$R_1 = F(\tau) \quad . \tag{2.97}$$

The asymptotic behavior as $\tau \uparrow \infty$ is given by equation (2.88).

The behavior as $\tau \downarrow 0$ can be obtained from (2.96):

$$V_1 \to 1 - \frac{1}{\pi_1} - \frac{2\pi_1}{\ln(1-\pi_1)} \tau^2 \tag{2.98a}$$

and

$$R_0 \to \left(-\frac{1}{\pi_1}\right)^{1/2} + \frac{(-\pi_1)^{3/2}}{\ln(1-\pi_1)} \tau^2 \quad . \tag{2.98b}$$

To study the stability of the surface $r = r_0(t)$, we use (2.2), (2.23), (2.95) and the prescribed values of P at the boundaries to get

$$\frac{V^{3/2}}{E} \frac{1}{\rho_0} \left.\frac{\partial P}{\partial r}\right|_{r_0(t)}$$

$$= \frac{1}{R_0 \ln \frac{R_1}{R_0}} \left\{ \frac{\pi_1}{R_1^2 \ln \frac{R_1^2}{R_0^2}} - \left[\frac{1}{R_0^2} - \left(\frac{1}{R_0^2} - \frac{1}{R_1^2}\right) \frac{1}{\ln \frac{R_1^2}{R_0^2}}\right]\right\} \tag{2.99}$$

and this is < 0 throughout the flow. Thus, we get Taylor instability at $r = r_0(t)$.

To solve the general initial-value problem, we suppose that we are given $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$. We use (2.24), (2.25), (2.75), the equation $E = -\epsilon$, (2.27), and (2.94) to compute V, E, t^*, and π_1. We find $\tau_0 \in [0,\infty)$

so that

$$R_1(0) = F(\tau_0) \ .$$

Then, if $\dot{r}_1(0) > 0$, the flow for $t \geq 0$ is given by

$$r_1(t) = V^{1/2} F\left(\tau_0 + \frac{t}{t^*}\right) . \tag{2.100}$$

If $\dot{r}_1(0) < 0$, the flow for $0 \leq t \leq t^* \tau_0$ is

$$r_1(t) = V^{1/2} F\left(\tau_0 - \frac{t}{t^*}\right) , \tag{2.101a}$$

and for $t \geq t^* \tau_0$

$$r_1(t) = V^{1/2} F\left(\frac{t}{t^*} - \tau_0\right) . \tag{2.101b}$$

TWO-DIMENSIONAL FLOW WITH CIRCULATION, ZERO PRESSURE ON THE FREE SURFACES

In place of the velocity potential (2.23), we have

$$\varphi = A(t) \ \ell n \frac{r}{r_1(t)} + C\,\theta + B(t) , \tag{2.102}$$

where θ is the azimuthal angle. The important case is where C is independent of time as well as space. The kinematical boundary conditions are still given by (2.24). The dynamical boundary conditions are obtained from (2.2), (2.102), and the fact that $P = 0$ on the free boundaries:

$$\dot{A} \ \ell n \frac{r_0}{r_1} - \frac{A}{r_1} \dot{r}_1 + \dot{B} + \frac{1}{2} \frac{A^2 + C^2}{r_0^2} = 0 , \tag{2.103a}$$

$$- \frac{A}{r_1} \dot{r}_1 + \dot{B} + \frac{1}{2} \frac{A^2 + C^2}{r_1^2} = 0 . \tag{2.103b}$$

From the kinematical conditions, conservation of volume follows as in (2.25). For the conservation of energy, we get

$$(A^2(t)+C^2)\ \ell n\ \frac{r_1(t)}{r_0(t)} = (A^2(0)+C^2)\ \ell n\ \frac{r_1(0)}{r_0(0)} = E\ . \tag{2.104}$$

We may use (2.27), (2.28), and (2.29d) to define t^*, τ, R_0, R_1, α, and V_1. Further, we let

$$\gamma = \frac{C}{E^{1/2}}\ . \tag{2.105}$$

The governing equations are (2.29a), (2.29c), and

$$(\alpha^2+\gamma^2)\ \ell n\ \frac{R_1}{R_0} = 1\ . \tag{2.106}$$

Since γ is constant, it is clear from (2.106) that we cannot have $R_0 = 0$. Thus, these flows will never exhibit "collisions". The minimum possible value of R_0 will be achieved when $\dot{r}_0 = A = \alpha = 0$:

$$\gamma^2\ \ell n\ \frac{R_1}{R_0} = 1 \Rightarrow V_1 = \frac{1}{1-e^{-2/\gamma^2}}\ . \tag{2.107}$$

Let us consider the flow for which $V_1(0)$ is given by (2.107). Then we get

$$\tau = \frac{1}{2\sqrt{2}} \int_{\frac{1}{1-e^{-2/\gamma^2}}}^{V_1} \left(\frac{\ell n\left(1+\frac{1}{\xi-1}\right)}{1-\frac{\gamma^2}{2}\ \ell n\left(1+\frac{1}{\xi-1}\right)} \right)^{1/2} d\xi\ . \tag{2.108}$$

In this equation we take the positive sign for the square root, and thus we get R_1 as a monotonically increasing function of τ:

$$R_1 = F(\tau) \quad . \tag{2.109}$$

From (2.108) we may obtain the behavior as $\tau \downarrow 0$:

$$V_1 \rightarrow \frac{1}{1-e^{-2/\gamma^2}} + \frac{\gamma^4}{2}\left(e^{\frac{1}{\gamma^2}} - e^{-\frac{1}{\gamma^2}}\right)^2 \tau^2 \quad , \tag{2.110a}$$

$$R_0 \rightarrow e^{-\frac{1}{2\gamma^2}}\left(e^{\frac{1}{\gamma^2}} - e^{-\frac{1}{\gamma^2}}\right)^{-1/2} + \frac{\gamma^4}{4}\, e^{\frac{1}{2\gamma^2}}\left(e^{\frac{1}{\gamma^2}} - e^{-\frac{1}{\gamma^2}}\right)^{5/2} \tau^2 \quad . \tag{2.110b}$$

The asymptotic behavior as $\tau \uparrow \infty$ is given by (2.34).

Now, suppose we have $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$. From (2.24), (2.25), (2.104), (2.105), and (2.27) we find V, E, γ and t^*. We find $\tau_0 \in [0,\infty)$ such that

$$R_1(0) = F(\tau_0) \; .$$

If $\dot{r}_1(0) > 0$, the solution for $t \geq 0$ is given by

$$r_1(t) = V^{1/2}\, F\left(\tau_0 + \frac{t}{t^*}\right) \quad . \tag{2.111}$$

If $\dot{r}_1(0) < 0$, the solution for $0 \leq t \leq t^* \tau_0$ is

$$r_1(t) = V^{1/2}\, F\left(\tau_0 - \frac{t}{t^*}\right) \tag{2.112a}$$

and for $t \geq t^* \tau_0$

$$r_1(t) = V^{1/2}\, F\left(\frac{t}{t^*} - \tau_0\right) \quad . \tag{2.112b}$$

TWO-DIMENSIONAL FLOW WITH CIRCULATION, AND PRESSURE APPLIED AT ONE OF THE FREE SURFACES

We treat now the case where there is circulation, and, as well, a pressure P_0, which is constant over the surface, is applied at the inner boundary. The pressure at the outer boundary is taken to be 0.

The velocity potential is given by (2.102). The kinematical boundary conditions and conservation of volume are given by (2.24) and (2.25). For the dynamical boundary conditions, we insert (2.102) into the Bernoulli equation (2.2) and get

$$\dot{A}\ \ell n \frac{r_0}{r_1} - \frac{A}{r_1}\ \dot{r}_1 + \dot{B} + \frac{1}{2}\frac{A^2+C^2}{r_0^2} + \frac{P_0}{\rho_0} = 0\ , \tag{2.113a}$$

$$-\frac{A}{r_1}\ \dot{r}_1 + \dot{B} + \frac{1}{2}\frac{A^2+C^2}{r_1^2} = 0\ . \tag{2.113b}$$

The expression of conservation of energy assumes the form

$$(A^2(t)+C^2)\ \ell n \frac{r_1(t)}{r_0(t)} - \frac{P_0}{\rho_0}\ r_0^2(t) = (A^2(0)+C^2)\ \ell n \frac{r_1(0)}{r_0(0)} - \frac{P_0}{\rho_0}\ r_0^2(0) \equiv \epsilon\ . \tag{2.114}$$

If $P_0 < 0$, ϵ will be > 0. There will also be cases when $\epsilon > 0$ and $P_0 > 0$. So first we consider $\epsilon > 0$ and set $E = \epsilon$. We define t^*, τ, R_0, R_1, α, γ, and V_1 by (2.27), (2.28), (2.105), and (2.29d). Let

$$\pi_0 = \frac{V}{E}\ \frac{P_0}{\rho_0}\ . \tag{2.115}$$

The equations governing the flow are (2.29a), (2.29c), and

$$(\alpha^2+\gamma^2)\ \ell n \frac{R_1}{R_0} = 1 + \pi_0(R_1^2-1)\ . \tag{2.116}$$

Since γ is constant, we cannot have $R_0 = 0$, and there will be no "collisions". The minimum value of R_0 will be achieved when $\alpha = 0$:

$$\gamma^2 \ \ell n \frac{R_1}{R_0} = 1 + \pi_0 \ R_0^2 \quad . \tag{2.117}$$

When $\pi_0 \geq 0$ there will be one and only one root $R_0 > 0$ of (2.117). We call this root $R_0(\gamma,\pi_0)$. For this case, the flow will have $r_0(t) \geq V^{1/2} R_0(\gamma,\pi_0)$. Given γ and $\pi_0 < 0$, there may be 0, 1, or 2 roots $R_0 > 0$ of (2.117). If there are no roots,

$$\gamma^2 \ \ell n \frac{R_1}{R_0} > 1 + \pi_0 \ R_0^2$$

for all $R_0 > 0$. On the other hand, from (2.116), for any real flow

$$\gamma^2 \ \ell n \ \frac{R_1}{R_0} \leq 1 + \pi_0 \ R_0^2 \ ,$$

and thus for such flows equation (2.117) will have 1 or 2 roots. We easily check that if (2.117) has just one root, we must have

$$\frac{\gamma^2}{2} \ \ell n \left(1 + \frac{1}{R_0^2(0)}\right) = 1 + \pi_0 \ R_0^2(0) \tag{2.118a}$$

and

$$- \frac{\gamma^2}{2R_0^2(0) \ R_1^2(0)} = \pi_0 \quad , \tag{2.118b}$$

from which we get

$$A(0) = 0 \tag{2.119a}$$

and, on combining (2.113a) and (2.113b),

$$\dot{A}(0) = 0 \ . \tag{2.119b}$$

In this case, the flow is steady, and the negative pressure P_0 on the inner boundary is just sufficient to balance the centrifugal force of the swirling motion. So we may consider the case where (2.117) has two distinct positive roots. We call them $R_0^{(1)}(\gamma,\pi_0)$ and $R_0^{(2)}(\gamma,\pi_0)$, with $R_0^{(1)}$ the smaller root. Thus, the flow will satisfy

$$V^{1/2}\, R_0^{(1)}(\gamma,\pi_0) \le r_0(t) \le V^{1/2}\, R_0^{(2)}(\gamma,\pi_0) \tag{2.120}$$

for all $t \ge 0$.

A more complete description of the restrictions imposed on $-\pi_0$ and γ^2 by the requirement that (2.117) have at least one solution is as follows: We must have, for $0 < -\pi_0 < \infty$ and $0 < \gamma^2 < \infty$,

$$0 < \gamma^2 \le G(-\pi_0) \tag{2.121a}$$

and

$$0 < -\pi_0 \le G^{-1}(\gamma^2)\ , \tag{2.121b}$$

where the curve $\gamma^2 = G(-\pi_0)$ or $-\pi_0 = G^{-1}(\gamma^2)$ is given parametrically by the equations

$$-\pi_0 = \frac{1}{\xi\left[(1+\xi)\ \ln\left(1+\frac{1}{\xi}\right) + 1\right]} \tag{2.122a}$$

and

$$\frac{\gamma^2}{2} = \frac{1}{\ln\left(1+\frac{1}{\xi}\right) + \frac{1}{1+\xi}} \tag{2.122b}$$

for $0 < \xi < \infty$. We see that

$$\frac{d}{d\xi}(-\pi_0) < 0, \qquad \frac{d}{d\xi}\left(\frac{\gamma^2}{2}\right) > 0\ ,$$

and hence

$$\frac{d\,G(x)}{dx} < 0, \qquad \frac{d\,G^{-1}(x)}{dx} < 0, \qquad 0 < x < \infty \,. \tag{2.123}$$

From (2.122) we obtain

$$G(x) \to \frac{1}{2x} \quad \text{as} \quad x \downarrow 0 \tag{2.124a}$$

and

$$G(x) \to \frac{2}{\ln x} \quad \text{as} \quad x \uparrow \infty\,; \tag{2.124b}$$

also

$$G^{-1}(x) \to \frac{x}{2}\, e^{2/x} \quad \text{as} \quad x \downarrow 0 \tag{2.125a}$$

and

$$G^{-1}(x) \to \frac{1}{2x} \quad \text{as} \quad x \uparrow \infty\,. \tag{2.125b}$$

Now for $\pi_0 < 0$, let us consider the special flow with $V_1(0) = (R_0^{(1)})^2 + 1$. We get

$$\tau = \frac{1}{2\sqrt{2}} \int_{(R_0^{(1)})^2+1}^{V_1} \left(\frac{\ln\left(1+\frac{1}{\xi-1}\right)}{1 + \pi_0(\xi-1) - \frac{\gamma^2}{2}\ln\left(1+\frac{1}{\xi-1}\right)} \right)^{1/2} d\xi\,. \tag{2.126}$$

Taking the positive square root, we see that (2.126) gives R_1 as a monotonically increasing function of τ for $0 < \tau < \tau^*$,

$$R_1 = F(\tau)\,, \tag{2.127}$$

where

$$\tau^* = \frac{1}{2\sqrt{2}} \int_{(R_0^{(1)})^2+1}^{(R_0^{(2)})^2+1} \left(\frac{\ln\left(1+\frac{1}{\xi-1}\right)}{1 + \pi_0(\xi-1) - \frac{\gamma^2}{2}\ln\left(1+\frac{1}{\xi-1}\right)} \right)^{1/2} d\xi\,. \tag{2.128}$$

As $\tau \downarrow 0$, we have

$$V_1 \to (R_0^{(1)})^2 + 1 + \frac{\gamma^2}{1 + \pi_0 (R_0^{(1)})^2}\left[\pi_0 + \frac{\gamma^2}{2}\left(\frac{1}{(R_0^{(1)})^2} - \frac{1}{(R_0^{(1)})^2+1}\right)\right]\tau^2 \tag{2.129a}$$

and

$$R_0 \to R_0^{(1)} + \frac{\gamma^2}{2R_0^{(1)}(1+\pi_0(R_0^{(1)})^2)}\left[\pi_0 + \frac{\gamma^2}{2}\left(\frac{1}{(R_0^{(1)})^2} - \frac{1}{(R_0^{(1)})^2+1}\right)\right]\tau^2 . \tag{2.129b}$$

At the other end of the range, we find the behavior as $(\tau^*-\tau) \downarrow 0$:

$$V_1 \to (R_0^{(2)})^2 + 1 - \frac{\gamma^2}{1+\pi_0(R_0^{(2)})^2}\left[-\pi_0 - \frac{\gamma^2}{2}\left(\frac{1}{(R_0^{(2)})^2} - \frac{1}{(R_0^{(2)})^2+1}\right)\right](\tau^*-\tau)^2 \tag{2.130a}$$

and

$$R_0 \to R_0^{(2)} - \frac{\gamma^2}{2R_0^{(2)}(1+\pi_0(R_0^{(2)})^2)}\left[-\pi_0 - \frac{\gamma^2}{2}\left(\frac{1}{(R_0^{(2)})^2} - \frac{1}{(R_0^{(2)})^2+1}\right)\right](\tau^*-\tau)^2 . \tag{2.130b}$$

If we get Taylor instability, it will occur at the surface $r = r_1(t)$. From (2.2), (2.102), (2.116), and the known values of P on the free boundaries, we find

$$\frac{V^{3/2}}{E}\frac{1}{\rho_0}\left.\frac{\partial P}{\partial r}\right|_{r_1(t)} = \frac{1}{R_1 \ln \frac{R_1}{R_0}}\left\{-\pi_0 + [1+\pi_0(R_1^2-1)]\left(\frac{1}{R_1^2} - \frac{\frac{1}{R_0^2} - \frac{1}{R_1^2}}{\ln \frac{R_1^2}{R_0^2}}\right)\right\}. \tag{2.131}$$

As in the case where there was no circulation, there is a number, R_0^*, dependent only on $-\pi_0$, $0 < R_0^{*2} < -\frac{1}{\pi_0}$, such that the expression in (2.131) is negative for $R_0 < R_0^*$ and positive for $R_0 > R_0^*$. R_0^* satisfies

$$\pi_0 = \frac{\ln\left(1+\frac{1}{R_0^{*2}}\right) - \frac{1}{R_0^{*2}}}{\ln\left(1+\frac{1}{R_0^{*2}}\right) + 1} . \tag{2.132}$$

The question of Taylor instability thus depends on the relation of $R_o^*(-\pi_o)$ to $R_o^{(1)}(\gamma,\pi_o)$ and $R_o^{(2)}(\gamma,\pi_o)$. We see from (2.117) that as $\gamma^2 \downarrow 0$ we get

$$R_o^{(1)}(\gamma,\pi_o) \to e^{-\frac{1}{2\gamma^2}}\left(e^{\frac{1}{\gamma^2}} - e^{-\frac{1}{\gamma^2}}\right)^{-1/2} \to 0\ , \tag{2.133}$$

as in (2.110b). And

$$R_o^{(2)}(\gamma,\pi_o) \to \left(-\frac{1}{\pi_o}\right)^{1/2} . \tag{2.134}$$

We recall that the expression

$$\frac{\gamma^2}{2}\ \ell n\left(1+\frac{1}{\xi}\right) - 1 - \pi_o\ \xi$$

vanishes when ξ is $(R_o^{(1)})^2$ or $(R_o^{(2)})^2$ and is negative for $(R_o^{(1)})^2 < \xi < (R_o^{(2)})^2$. Hence, differentiating with respect to ξ at $(R_o^{(1)})^2$ and $(R_o^{(2)})^2$, we get that

$$-\frac{\gamma^2}{2\xi(1+\xi)} - \pi_o$$

is negative at $(R_o^{(1)})^2$ and positive at $(R_o^{(2)})^2$. Now differentiate (2.117) with respect to γ^2 when R_o is $R_o^{(1)}$ or $R_o^{(2)}$. We find

$$\frac{\partial R_o^{(1)}(\gamma,\pi_o)}{\partial\gamma} \geq 0 \tag{2.135a}$$

and

$$\frac{\partial R_o^{(2)}(\gamma,\pi_o)}{\partial\gamma} \leq 0\ . \tag{2.135b}$$

So, as γ^2 increases from 0 to $G(-\pi_0)$, the interval $\left(R_0^{(1)}, R_0^{(2)}\right)$ shrinks from $\left(0, \left(-\frac{1}{\pi_0}\right)^{1/2}\right)$ to a small neighborhood of $\tilde{R}_0$, where $\tilde{R}_0$ satisfies (2.118) with γ^2 replaced by $G(-\pi_0)$. It is easy to see that

$$\tilde{R}_0 > R_0^* \quad . \tag{2.136}$$

For, as we saw in (2.119), the corresponding limiting flow is stationary, and the only acceleration of fluid elements at $r = V^{1/2} (\tilde{R}_0^2+1)^{1/2}$ is the centripetal acceleration inward, which can only be caused by a positive value of $\frac{\partial P}{\partial r}$ at that point. As γ^2 increases from 0 to $G(-\pi_0)$, it will cross a value $H(-\pi_0)$ such that

$$R_0^{(1)} \left((H(-\pi_0))^{1/2}, \pi_0 \right) = R_0^* \quad . \tag{2.137}$$

For $0 < \gamma^2 < H(-\pi_0)$, the surface $r = r_1(t)$ will be stable when $V^{1/2} R_0^{(1)}(\gamma,\pi_0) \le r_0(t) \le V^{1/2} R_0^*$ and unstable when $V^{1/2} R_0^* < r_0(t) \le V^{1/2} R_0^{(2)}(\gamma,\pi_0)$. For $H(-\pi_0) < \gamma^2 < G(-\pi_0)$, the surface $r = r_1(t)$ will be unstable throughout the motion. We may compute $H(-\pi_0)$ from (2.137), (2.132), and (2.117). The result is

$$H = 2\,(1-\pi_0)\, R_0^{*2} \quad . \tag{2.138}$$

For the general initial-value problem, with $P_0 < 0$, let us suppose we have given $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$. We use (2.24), (2.25), (2.114), the equation $E = \epsilon$, (2.105), (2.27), (2.115), and (2.128) to find V, E, γ, t^*, π_0, and τ^*. Using (2.127), we find $\tau_0 \in [0,\tau^*]$ such that

$$R_1(0) = F(\tau_0) \quad .$$

If $\dot{r}_1(0) \geq 0$, the flow for $0 \leq t \leq t^* (\tau^* - \tau_0)$ is given by

$$r_1(t) = V^{1/2} F\left(\tau_0 + \frac{t}{t^*}\right) ; \qquad (2.139a)$$

for $t^*(\tau^* - \tau_0) \leq t \leq t^* (2\tau^* - \tau_0)$, by

$$r_1(t) = V^{1/2} F\left(2\tau^* - \tau_0 - \frac{t}{t^*}\right) ; \qquad (2.139b)$$

for $t^* (2\tau^* - \tau_0) \leq t \leq t^* (3\tau^* - \tau_0)$, by

$$r_1(t) = V^{1/2} F\left(\frac{t}{t^*} - 2\tau^* + \tau_0\right) ; \qquad (2.139c)$$

and in general, by

$$r_1(t) = r_1(t + 2t^* \tau^*) . \qquad (2.139d)$$

If $\dot{r}_1(0) \leq 0$, for $0 \leq t \leq t^* \tau_0$ we have

$$r_1(t) = V^{1/2} F\left(\tau_0 - \frac{t}{t^*}\right) ; \qquad (2.140a)$$

for $t^* \tau_0 \leq t \leq t^* (\tau_0 + \tau^*)$,

$$r_1(t) = V^{1/2} F\left(\frac{t}{t^*} - \tau_0\right) ; \qquad (2.140b)$$

for $t^* (\tau_0 + \tau^*) \leq t \leq t^* (\tau_0 + 2\tau^*)$,

$$r_1(t) = V^{1/2} F\left(2\tau^* + \tau_0 - \frac{t}{t^*}\right) ; \qquad (2.140c)$$

and, in general, we have the periodicity (2.139d).

Suppose now that ϵ given by (2.114) is > 0 but $P_0 > 0$. As we have seen, there is one positive root $R_0(\gamma,\pi_0)$ of (2.117). Consider the special flow with $V_1(0) = (R_0(\gamma,\pi_0))^2 + 1$. Then

$$\tau = \frac{1}{2\sqrt{2}} \int_{(R_0(\gamma,\pi_0))^2+1}^{V_1} \left(\frac{\ln\left(1+\frac{1}{\xi-1}\right)}{1+\pi_0(\xi-1) - \frac{\gamma^2}{2}\ln\left(1+\frac{1}{\xi-1}\right)} \right)^{1/2} d\xi \quad . \tag{2.141}$$

If we take the positive square root here, we get R_1 as a monotonically increasing function of τ for $\tau > 0$:

$$R_1 = F(\tau) \quad . \tag{2.142}$$

The behavior as $\tau \downarrow 0$ is

$$V_1 \rightarrow (R_0(\gamma,\pi_0))^2+1 + \frac{\gamma^2\left(\pi_0 + \frac{\gamma^2}{2(R_0(\gamma,\pi_0))^2((R_0(\gamma,\pi_0))^2+1)}\right)}{1 + \pi_0(R_0(\gamma,\pi_0))^2} \tau^2 \quad , \tag{2.143a}$$

$$R_0 \rightarrow R_0(\gamma,\pi_0) + \frac{\gamma^2 \left(\pi_0 + \frac{\gamma^2}{2(R_0(\gamma,\pi_0))^2((R_0(\gamma,\pi_0))^2+1)}\right)}{2R_0(\gamma,\pi_0)(1+\pi_0(R_0(\gamma,\pi_0))^2)} \tau^2 \quad . \tag{2.143b}$$

As $\tau \uparrow \infty$ we get

$$R_1 \rightarrow e^{\sqrt{2\pi_0}\,\tau} \quad . \tag{2.144}$$

Regarding Taylor instability, it will occur only at the inner boundary, $r = r_0(t)$. From (2.2), (2.102), (2.116), and the known boundary values of P, we find

$$\frac{V^{3/2}}{E}\frac{1}{\rho_0}\frac{\partial P}{\partial r}\bigg|_{r_0(t)} = \frac{1}{R_0 \ell n \frac{R_1}{R_0}}\left\{-\pi_0 + (1+\pi_0(R_1^2-1))\left(\frac{1}{R_0^2} - \frac{\left(\frac{1}{R_0^2}-\frac{1}{R_1^2}\right)}{\ell n \frac{R_1^2}{R_0^2}}\right)\right\} . \quad (2.145)$$

As in the case without circulation, we find a unique R_0^+, dependent only on π_0, with $0 < R_0^+ < \infty$ and

$$\left(1+\frac{1}{R_0^{+2}}\right)\ell n\left(1+\frac{1}{R_0^{+2}}\right) - \frac{1}{R_0^{+2}} = \pi_0 . \quad (2.146)$$

When $r_0(t) \le V^{1/2} R_0^+$, we get no Taylor instability. When $r_0(t) > V^{1/2} R_0^+$, we get Taylor instability at $r = r_0(t)$.

So the only question is the relation of R_0^+ to the lower limit $R_0(\gamma,\pi_0)$ of the inner dimensionless radius. It is easy to see from (2.117) that we have

$$R_0(\gamma,\pi_0) \downarrow 0 \quad \text{as} \quad \gamma^2 \downarrow 0 \quad (2.147a)$$

and

$$R_0(\gamma,\pi_0) \uparrow \infty \quad \text{as} \quad \gamma^2 \uparrow \infty . \quad (2.147b)$$

As γ^2 increases from 0 to ∞, it will cross a value $K(\pi_0)$ such that

$$R_0(K(\pi_0),\pi_0) = R_0^+ . \quad (2.148)$$

From (2.148), (2.117), and (2.146), we get

$$K = 2(1+R_0^{+2}) . \quad (2.149)$$

When $\gamma^2 > K(\pi_0)$, the surface $r = r_0(t)$ will be unstable during the whole motion.

Now consider the general initial-value problem, where we have $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$. From (2.24), (2.25), (2.114), the equation $E = \epsilon$, (2.27), (2.105), and (2.115) we compute V, E, t^*, γ, and Π_0. We find $\tau_0 \in [0,\infty)$ such that

$$R_1(0) = F(\tau_0) \quad .$$

If $\dot{r}_1(0) \geq 0$, the flow for $t \geq 0$ is given by

$$r_1(t) = V^{1/2} \; F\left(\tau_0 + \frac{t}{t^*}\right) . \tag{2.150}$$

If $\dot{r}_1(0) \leq 0$, the flow for $0 \leq t \leq t^* \tau_0$ is

$$r_1(t) = V^{1/2} F\left(\tau_0 - \frac{t}{t^*}\right) \tag{2.151a}$$

and for $t \geq t^* \tau_0$

$$r_1(t) = V^{1/2} \; F\left(\frac{t}{t^*} - \tau_0\right) . \tag{2.151b}$$

Finally, we consider the case where ϵ as given by (2.114) is < 0. Write $E = -\epsilon$. Then we use (2.27), (2.28), (2.29d), and (2.105) to find t^*, τ, R_0, R_1, α, V_1 and γ and we set

$$\Pi_0 = \frac{V}{E} \; \frac{p_0}{\rho_0} \quad . \tag{2.152}$$

The governing equations are (2.29a), (2.29c), and

$$(\alpha^2+\gamma^2) \; \ell n \frac{R_1}{R_0} = - 1 + \Pi_0(R_1^2-1) \quad . \tag{2.153}$$

For this flow we must have $R_0 \geq \tilde{R}_0(\gamma,\pi_0)$, where $\tilde{R}_0$ is the unique positive solution of

$$\frac{\gamma^2}{2} \ln\left(1+\frac{1}{\tilde{R}_0^2}\right) = -1 + \pi_0 \tilde{R}_0^2 \quad . \tag{2.154}$$

It is clear that

$$\tilde{R}_0 \downarrow \left(\frac{1}{\pi_0}\right)^{1/2} \quad \text{when} \quad \gamma^2 \downarrow 0 \, , \tag{2.155a}$$

$$\tilde{R}_0 \uparrow \infty \quad \text{when} \quad \gamma^2 \uparrow \infty \, , \tag{2.155b}$$

and

$$\frac{d\tilde{R}_0}{d\gamma^2} \geq 0 \ . \tag{2.155c}$$

Consider the flow with $V_1(0) = (\tilde{R}_0(\gamma,\pi_0))^2 + 1$. We get

$$\tau = \frac{1}{2\sqrt{2}} \int_{\tilde{R}_0^2+1}^{V_1} \left(\frac{\ln\left(1+\frac{1}{\xi-1}\right)}{-1+\pi_0(\xi-1)-\frac{\gamma^2}{2}\ln\left(1+\frac{1}{\xi-1}\right)} \right)^{1/2} d\xi \quad . \tag{2.156}$$

This equation, with positive square roots, determines R_1 *as a monotonically* increasing function of τ for $\tau \in (0,\infty)$:

$$R_1 = F(\tau) \ . \tag{2.157}$$

As $\tau \downarrow 0$ we get

$$V_1 \to \tilde{R}_0^2 + 1 + \frac{2\left(\pi_0 + \frac{\gamma^2}{2\tilde{R}_0^2(\tilde{R}_0^2+1)}\right)}{\ln\left(1+\frac{1}{\tilde{R}_0^2}\right)} \tau^2 \quad , \tag{2.158a}$$

$$R_0 \to \tilde{R}_0 + \frac{\Pi_0 + \dfrac{\gamma^2}{2\tilde{R}_0^2(\tilde{R}_0^2+1)}}{\tilde{R}_0\ \ell n\left(1+\dfrac{1}{\tilde{R}_0^2}\right)}\ \tau^2 \quad . \tag{2.158b}$$

As $\tau \uparrow \infty$ we have equation (2.144).

The Taylor instability at the surface $r = r_0(t)$ is checked by using (2.2), (2.102), (2.153), and the given values of P at the inner and outer surfaces:

$$\frac{V^{3/2}}{E}\ \frac{1}{\rho_0}\ \frac{\partial P}{\partial r}\bigg|_{r_0(t)} = \frac{1}{R_0 \ell n \dfrac{R_1}{R_0}}\left\{\frac{-\Pi_0}{R_1^2\ \ell n\ \dfrac{R_1^2}{R_0^2}} - \left[\frac{1}{R_0^2} - \frac{\left(\dfrac{1}{R_0^2} - \dfrac{1}{R_1^2}\right)}{\ell n\ \dfrac{R_1^2}{R_0^2}}\right]\right\} \quad , \tag{2.159}$$

and this is negative, giving us instability at $r = r_0(t)$, throughout the flow.

For the general initial-value problem we use $r_0(0)$, $r_1(0)$, and $\dot{r}_1(0)$ to compute V, E, t^*, γ, and Π_0 from (2.24), (2.25), (2.114), the equation $E = -\epsilon$, (2.27), (2.105), and (2.152). From (2.157) we find $\tau_0 \in [0,\infty)$ such that

$$R_1(0) = F(\tau_0) \ .$$

If $\dot{r}_1(0) \geq 0$, the flow for $t \geq 0$ is

$$r_1(t) = V^{1/2}\ F\left(\tau_0 + \frac{t}{t^*}\right) \quad . \tag{2.160}$$

If $\dot{r}_1(0) \leq 0$, the flow for $0 \leq t \leq t^*\ \tau_0$ is

$$r_1(t) = V^{1/2}\ F\left(\tau_0 - \frac{t}{t^*}\right) \ , \tag{2.161a}$$

and for $t \geq t^*\ \tau_0$,

$$r_1(t) = V^{1/2}\ F\left(\frac{t}{t^*} - \tau_0\right) \ . \tag{2.161b}$$

REFERENCES

1. Joel C. W. Rogers, "Relation of Turbulence to Energy Conservation, Hadamard Instabilities, and Stochastic Processes", fourth in a series of reports which will constitute part of the paper "Algorithms for Hyperbolic and Hydrodynamic Free Boundary Problems".

2. Joel C. W. Rogers, "Algorithm for Water Waves, Error Bound for Pressure Computation", second in series of reports referred to in previous reference.

3. Joel C. W. Rogers, "Water Waves: Analytic Solutions, Uniqueness and Continuous Dependence on the Data", Naval Ordnance Laboratory NSWC/WOL/TR 75-43 (1975).

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